RADIATIVE TRANSFER LECTURE IV

MULTIPLE SCATTERING — PART 2 — ACCURATE SOLUTIONS

by Jim Pollack

ACCURATE SOLUTIONS OF THE EQUATION OF RADIATIVE TRANSFER:
I. DOUBLING/ADDING (SOLAR PROBLEM)

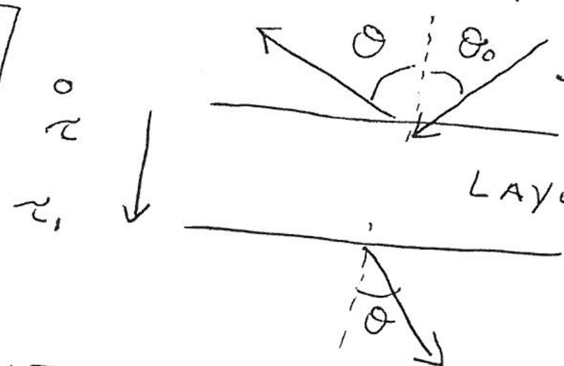
BASIC IDEA:

- DOUBLING (USED FOR HOMOGENEOUS LAYER – CONSTANT $\tilde{\omega}_0$, $P(\Theta)$)
  - BEGIN WITH VERY LOW OPTICAL DEPTH ($\tau \sim 10^{-8}$)
  - USE SINGLE SCATTERING TO DEFINE SCATTERING AND TRANSMISSION PROPERTIES – S & T
  - USE EQUATIONS FOR COMBINING 2 SUCH LAYER → S & T FOR LAYER WITH $2 * \tau$
  - KEEP DOUBLING UNTIL GET TO τ OF INTEREST
- WHY DOES THIS WORK?
  - ERRORS (DUE TO INITIAL SINGLE SCATTERING APPROXIMATION) GROW LINEARLY WITH # DOUBLINGS
  - OPTICAL DEPTH GROWS EXPONENTIALLY WITH # DOUBLING
- ADDING
  - COMBINING S & T FUNCTIONS OF TWO INHOMOGENEOUS LAYERS TO GET S & T FUNCTIONS OF COMBINED LAYERS

DOUBLING / ADDING PROCEDURE

(FOR INTENSITY ONLY —
WOULD INVOLVE MATRICES FOR POLARIZATI

— DEFINE S and T FUNCTIONS



transmitted $\measuredangle$

$\mu_{tran} = \cos(\pi - \theta)$
$= -\mu$
$\phi_{tran} = \phi_0$

Incident $\measuredangle$'s

$\mu_{inc} = \cos(\pi - \theta_0) = -\mu_0$
$\phi_{inc} = \phi_0$

reflected $\measuredangle$

$\mu_{ref} = \cos\theta = \mu$
$\phi_{ref} = \phi$

DIFFUSELY REFLECTED INTENSITY

$$\frac{I_{ref}}{F}(\tau = 0, \mu, \phi) \equiv \frac{1}{4\mu} S(\mu, \phi; \mu_0, \phi_0)$$

DIFFUSELY TRANSMITTED INTENSITY

$$\frac{I^{DIF}_{Tran}}{F}(\tau = \tau_1, \mu, \phi) = \frac{1}{4\mu} T(\mu, \phi; \mu_0, \phi_0).$$

TOTAL TRANSMITTED INTENSITY

$$\frac{I^{TOT}_{Tran}}{F}(\tau = \tau_1, \mu, \phi) = \frac{1}{4\mu} T(\mu, \phi; \mu_0, \phi_0)$$
$$+ \pi e^{-\tau_1/\mu_0} \delta(\mu + \mu_0) \delta(\phi - \phi_0)$$

ADVANTAGE OF S & T FUNCTIONS:

- SYMMETRIC

$$S(\mu, \phi; \mu_0, \phi_0) = S(\mu_0, \phi_0; \mu, \phi)$$
$$T(\mu, \phi; \mu_0, \phi_0) = T(\mu_0, \phi_0; \mu, \phi)$$

WHEN RADIATION IS INCIDENT IN ALL DIRECTIONS !

$$\frac{I}{F}_{ref}(\tau=0;\mu,\phi) = \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} S(\mu,\phi;\mu_0,\phi_0)\, I_{INC}(\mu_0,\phi_0)\, d\phi_0\, d\mu_0$$

$$\frac{I}{F}_{Tran}(\tau_1,\mu,\phi) = \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T(\mu,\phi;\mu_0,\phi_0)\, I_{INC}(\mu_0,\phi_0)\, d\phi_0\, d\mu_0$$

PROCEDURE

STEP 1 . DEFINE S + T FUNCTIONS FOR INITIAL, VERY OPTICALLY THIN LAYER:

$$S \cong \left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)^{-1} P(\mu,\phi;-\mu_0,\phi_0)\left[1-\exp\left[-\tau\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right]\right]$$

$$T \cong \left(\frac{1}{\mu}-\frac{1}{\mu_0}\right) P(-\mu,\phi;-\mu_0,\phi_0)\left[\exp\left(-\frac{\tau}{\mu_0}\right)-\exp\left(-\frac{\tau}{\mu}\right)\right]$$

STEP 2 . FOURIER ANALYZE S, T, I

$\longrightarrow$ reduce equations from ones involving $\underline{\underline{4}}$ ∡'s to $\underline{\underline{2}}$ ∡'s

(recall S etc depend only on $|\phi-\phi_0|$)

$$S = \sum_{m=0}^{\infty} S^m(\tau,\mu,\mu_0) \cos m(\phi-\phi_0)$$

$$T = \sum_{m=0}^{\infty} T^m(\tau,\mu,\mu_0) \cos m(\phi-\phi_0)$$

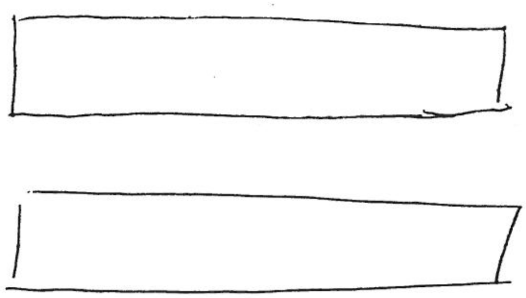
USE ORTHOGONALITY OF $\cos m(\phi - \phi_0)$ to DERIVE SEPARATE EQUATIONS FOR EACH FOURIER COEFFICIENT:

if $m \neq m'$ $\quad \int_0^{2\pi} \cos[m(\phi-\phi_0)] \cos[m'(\phi-\phi_0)]\, d\phi_0 = 0$

if $m = m'$ $\rightarrow \int = \pi$ for $m = m' \neq 0$

$\int = 2\pi$ for $m = m' = 0$

STEP 3 . FIND THE ~~COMBINED~~ S & T FUNCTIONS, S_c, T_c, OF COMBINING 2 LAYERS HAVING KNOWN S & T FUNCTIONS



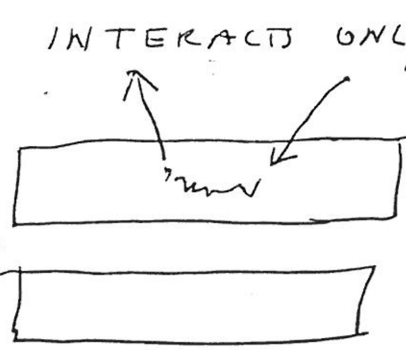
top layer
$S_\tau, T_\tau, \tau_\tau$

bottom Layer
S_b, T_b, τ_b

THIS IS DONE SEPARATELY FOR EACH FOURIER COMPONENT

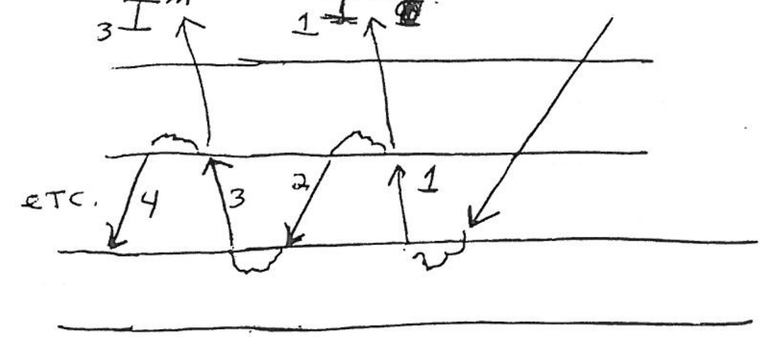
THERE ARE 5 COMPONENTS to THE DIFFUSELY REFLECTED LIGHT

1 LIGHT INTERACTS ONLY WITH TOP LAYER



$$\frac{I^m 1}{F} = \frac{1}{4\mu} S_\tau^m$$

(2) LIGHT IS <u>DIRECTLY</u> TRANSMITTED THRU TOP LAYER, INTERACTS MULITIPLE TIMES WITH BOTTOM & TOP LAYERS, WITH <u>DIRECT</u> TRANSMISSION BACK OUT THE TOP LAYER OCCURRING AFTER EACH UPWARD BOUNCE



$$\frac{I^m{}_2}{F} = \sum_{n \text{ odd}} {}_nI^m$$

AT POSITION 1 between Layers

$${}_1I^m = I^m(1)\, e^{-\tau/\mu}$$

$$= \frac{1}{4\mu} S_b^m(\mu,\mu_0)\, e^{-\tau/\mu} e^{-\tau/\mu_0}$$

$${}_3I^m = I^m(3)\, e^{-\tau/\mu}$$

$$I^m(3) = \frac{1}{4\pi\mu} f \int_0^1 S_b^m(\mu,\mu')\, I^m(2;\mu')\, d\mu'$$

from azimuthal integration

$$I^m(2) = \frac{1}{4\pi\mu'} f \int_0^1 \tilde{S}_\tau^m(\mu',\mu'')\, I^m(1;\mu'')\, d\mu''$$

$$I^m(1) = \frac{1}{4\mu''} S_b^m(\mu'',\mu_0)\, e^{-\tau/\mu_0}$$

reflection off bottom & top different in nomency.

$$\rightarrow I^m(2) = \frac{1}{4\mu'} e^{-\tau/\mu_0} \underbrace{\boxed{\frac{f}{4\pi}\int_0^1 \tilde{S}_\tau^m(\mu',\mu'')\, S_b^m(\mu'',\mu_0)\, \frac{d\mu''}{\mu''}}}_{{}_2S^m(\mu',\mu_0)}$$

$$\rightarrow I^m(3) = \frac{1}{4\mu} e^{-\tau/\mu_0}\, \frac{f}{4\pi}\int_0^1 S_b^m(\mu,\mu')\, {}_2S^m(\mu',\mu_0)\, \frac{d\mu'}{\mu'}$$

$$I^m(3) \overset{\text{or}}{=} \frac{1}{4\mu} e^{-\tau/\mu_0} \, {}_3S^m(\mu, \mu_0)$$

In general

n odd: $I^m(n; \mu) = \frac{1}{4\mu} e^{-\tau/\mu_0} \, {}_nS^m(\mu, \mu_0)$

where:

n odd, $n \neq 1$ $\quad {}_nS^m(\mu, \mu_0) = \frac{f}{4\pi} \int_0^1 S_b^m(\mu, \mu') \, {}_{(n-1)}\tilde{S}^m(\mu', \mu_0) \frac{d\mu'}{\mu'}$

n even $\quad {}_nS^m(\mu', \mu_0) = \frac{f}{4\pi} \int_0^1 \tilde{S}_\tau^m(\mu', \mu'') \, {}_{(n-1)}S^m(\mu'', \mu_0) \frac{d\mu''}{\mu''}$

n = 1 $\quad {}_1S^m(\mu, \mu_0) = S_b^m(\mu, \mu_0)$

thus

$$I^m 2 = \frac{1}{4\mu} \sum_{odd}(\mu, \mu_0) \, e^{-\tau/\mu} e^{-\tau/\mu_0}$$

$$\sum_{odd}(\mu, \mu_0) = \sum_{n\ odd} {}_nS^m(\mu, \mu_0)$$

NOTES

— ALL τ above are for top layer (τ_τ)

— $\tilde{S}_\tau^m$ is S FUNCTION FOR REFLECTION FROM BOTTOM OF TOP LAYER; it $= S_\tau^m$, REFECTION OFF THE TOP OF THE TOP LAYER, IF THE TOP LAYER IS HOMOGENEOUS, but $\tilde{S}_\tau^m \neq S_\tau^m$ IF THE TOP LAYER IS INHOMOGENEOUS

Can truncate at some point

— CAN SIMPLY THINK OF COMBINING 2 S FUNCTION AS A SIMPLE MULTIPLICATIO

$$S1 \cdot S2 \equiv \frac{f}{4\pi} \int_0^1 S1(\mu, \mu') \, S2(\mu', \mu_0) \, \frac{d\mu'}{\mu'}$$

MAKES DERIVATIONS QUICKER ~~QUICKER~~

BUT, IMPORTANT TO KEEP ORDER OF MULTIPLICATIONS STRAIGHT, SINCE

$$S1 \cdot S2 \neq S2 \cdot S1 \text{ IN MANY CASES}$$

USE ~~[illegible]~~ MULTIPLE BOUNCE PICTURE to DO THIS

3 LIGHT DIRECTLY TRANSMITTED THRU TOP LAYER, MULTIPLY BOUNCES BETWEEN 2 LAYERS, + IS DIFFUSELY TRANSMITTED

$$\vec{I^m 3}(\mu) = \frac{1}{4\mu} e^{-\tau/\mu_0} \frac{f}{4\pi} \int_0^1 T_\tau^m(\mu, \mu') \sum_{odd}(\mu', \mu_0) \frac{d\mu'}{\mu'},$$

$$\left(\text{or } \frac{1}{4\mu} e^{-\tau/\mu_0} \, T_\tau^m \cdot \sum_{odd}\right)$$

4 LIGHT DIFFUSELY TRANSMITTED THRU TOP LAYER, MULTIPLE BOUNCES BETWEEN LAYERS, DIRECT TRANSMISSION OUT

$$I^m 4(\mu) = \frac{1}{4\mu} e^{-\tau/\mu} \sum_{odd}^m \cdot T_\tau^m$$

5 DIFFUSE IN + OUT, MULTIPLE BOUNCES

$$I^m_{,} 5(\mu) = \frac{1}{4\mu} \, T_\tau^m \cdot \sum_{odd}^m \cdot T_\tau^m$$

J = ... top layer from below

$$\underset{\text{even}}{S_n} = \frac{1}{4\pi}\int_0^1\int_0^{2\pi} \underset{T}{\tilde{S}}\ \underset{\text{odd}}{S_n}\ \frac{d\mu'}{\mu'}\, d\phi'$$

$$\underset{\text{odd}}{S_n} = \frac{1}{4\pi}\int_0^1\int_0^{2\pi} \underset{B}{S}\ \underset{\text{even}}{S_n}\ \frac{d\mu'}{\mu'}\, d\phi'$$

$n \geq 3$

2/3/93

diffuse radiation may escape from the double layer

$$\frac{S(2\tau;\mu,\phi;\mu_0,\phi_0)F}{4\mu} = \frac{S(\tau;\mu,\phi;\mu_0,\phi_0)F}{4\mu} + e^{-\tau/\mu}\frac{1}{4\mu}\Sigma_0(\tau;\mu,\phi;\mu_0,\phi_0)Fe^{-\tau/\mu_0}$$

$$+\frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T(\tau;\mu,\phi;\mu',\phi')\frac{1}{4\mu'}\Sigma_0(\tau;\mu',\phi';\mu_0,\phi_0)d\mu'd\phi'Fe^{-\tau/\mu_0}$$

$$+e^{-\tau/\mu}\frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi}\Sigma_0(\tau;\mu,\phi;\mu',\phi')\frac{T(\tau;\mu',\phi';\mu_0,\phi_0)}{4\mu'}d\mu'd\phi'F \qquad (1)$$

$$+\frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} T(\tau;\mu,\phi;\mu'',\phi'')\frac{1}{4\pi\mu''}\Sigma_0(\tau;\mu'',\phi'';\mu',\phi')$$

$$\times\frac{T(\tau;\mu',\phi';\mu_0,\phi_0)}{4\mu'}d\mu'd\phi'd\mu''d\phi''F.$$

where

$$\Sigma_0(\tau;\mu,\phi;\mu_0,\phi_0) \equiv \sum_{n=1,3,\ldots}^{\infty} S_n(\tau;\mu,\phi;\mu_0,\phi_0), \qquad (2)$$

$$S_1(\tau;\mu,\phi;\mu_0,\phi_0) \equiv \underset{B}{S}(\tau;\mu,\phi;\mu_0,\phi_0), \qquad (3)$$

and

$$S_n(\tau;\mu,\phi;\mu_0,\phi_0) \equiv \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau;\mu,\phi;\mu',\phi')S_{n-1}(\tau;\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi'. \qquad (4)$$

The factor $F/4\mu$ is included in equation (1) to make clear the physical basis for that equation. The first term on the right-hand side is the intensity of radiation diffusely scattered from the upper layer without interaction with the lower layer. The remaining terms are the radiation transmitted downward by the upper layer (diffusely or without interaction), scattered any number of times back and forth between the two layers, and finally transmitted upward by the upper layer. Similarly an equation for $T(2\tau;\mu,\phi;\mu_0,\phi_0)$ is

$$T(2\tau;\mu,\phi;\mu_0,\phi_0) = \underset{B}{T}(\tau;\mu,\phi;\mu_0,\phi_0)e^{-\tau/\mu_0} + e^{-\tau/\mu}\underset{T}{T}(\tau;\mu,\phi;\mu_0,\phi_0)$$

$$+e^{-\tau/\mu}\Sigma_e(\tau;\mu,\phi;\mu_0,\phi_0)e^{-\tau/\mu_0}$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\underset{B}{T}(\tau;\mu,\phi;\mu',\phi')\underset{T}{T}(\tau;\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi'$$

$$+e^{-\tau/\mu_0}\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\underset{B}{T}(\tau;\mu,\phi;\mu',\phi')\Sigma_e(\tau;\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi' \qquad (5)$$

$$+e^{-\tau/\mu}\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\Sigma_e(\tau;\mu,\phi;\mu',\phi')\underset{T}{T}(\tau;\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi'$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}\underset{B}{T}(\tau;\mu,\phi;\mu'',\phi'')\Sigma_e(\tau;\mu'',\phi'';\mu',\phi')$$

$$\times\underset{T}{T}(\tau;\mu',\phi';\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi'\frac{d\mu''}{\mu''}d\phi''.$$

where

$\Sigma_e(\tau;$

The above equations are eq[...] derived equations which inc[...] sources (e.g., thermal emissi[...] doubling principle.

III.

The above equations pro[...] increasing thickness provide[...] accurate solutions of the ra[...] inaccuracies can lead to an i[...] the radiation singly scatter[...] phase function. For $\tau \ll 1$ [...] radiation is $\simeq\tau$, and hence [...] initial optical thickness $\tau_0 =$ [...] tive error will not increase [...] for $\tau \ll 1$, S and T are $\ll 1$) [...] if the computations are star[...] scattering and transmission [...] carried to very thick layers.

The well-known expression [...] khar 1960) gives us the follow[...]

$S(\tau_0;\mu,\phi;\mu_0,\phi_0) \simeq \Big(\frac{1}{\mu} + \frac{}{\mu}$ [...]

$T(\tau_0;\mu,\phi;\mu_0,\phi_0) \simeq \Big(\frac{1}{\mu} - \frac{1}{\mu_0}\Big)$ [...]

where $P(\mu,\phi;\mu_0,\phi_0)$ is the ph[...] tering. These equations prov[...] brackets involve the differen[...] lost on a computer; hence it [...] keep terms through τ_0^2.

The sums (eq. [2] and eq. [...] terms replaced by the geomet[...] tered back and forth between [...] a number of such scatterings, [...] approaches a constant value [...] thick layers).

Considerable advantage is [...] tions are expanded in Fourier [...] in cosines of the scattering an[...]

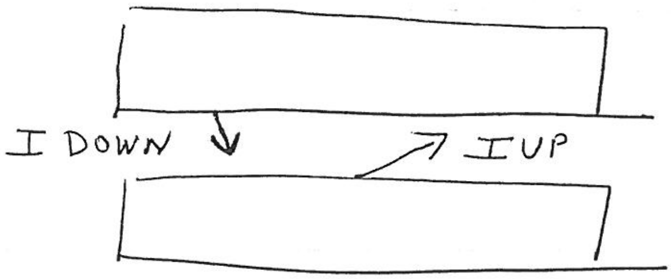
$S(\tau;\mu,\phi;\mu_0$ [...]

[wh]ere

$S^m(\tau;\mu;\mu_0) = \frac{}{(2-}$ [...]

$\delta_{0,m} =$

INTERNAL RADIATION FIELDS
(BETWEEN LAYERS)



I UP — SUM ALL ODD $I^m(n)$

I DOWN — SUM ALL EVEN $I^m(n)$

+ LIGHT DIFFUSELY TRANSMITTED THRU TOP LAYER

$$\rightarrow \quad I^m_{UP} = \frac{1}{4\mu} \sum^m_{odd}(\mu,\mu_0)\, e^{-\tau/\mu_0}$$ direct in

$$+ \frac{1}{4\mu} \sum^m_{odd} \cdot T^m_\tau$$ diffuse in

$$I^m_{DOWN} = \frac{1}{4\mu} \sum^m_{even}(\mu,\mu_0)\, e^{-\tau/\mu_0}$$

$$+ \frac{1}{4\mu} \sum^m_{even} \cdot T^m_\tau$$

$$+ \frac{1}{4\mu} T^m_\tau(\mu,\mu_0)$$

PRACTICAL TIPS ON ADDING/DOUBLING

— MINIMIZE # BOUNCES NEED TO EXPLICITLY EVALUATE TO FIND $\sum_{odd}$, $\sum_{even}$

- TEST TO SEE IF RADIATION FIELD HAS BECOME ISOTROPIC

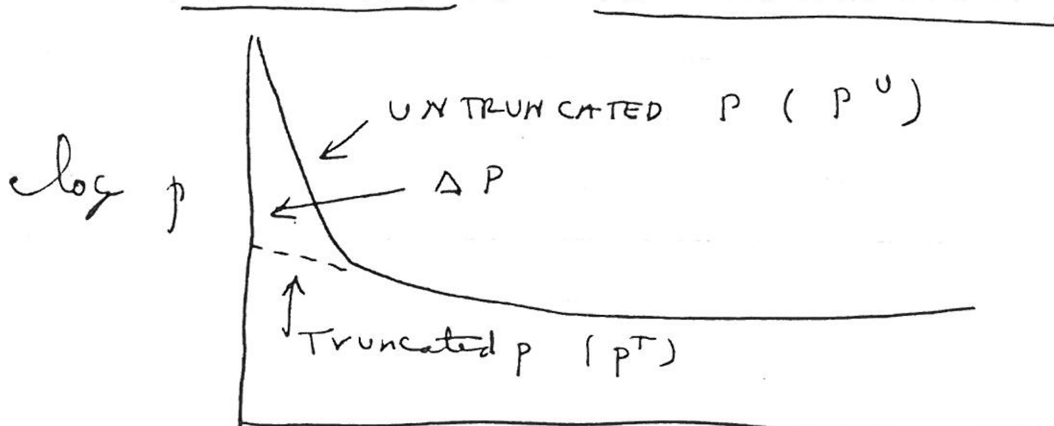
$$I^m(N+2)(\mu) = \lambda I^m(N)(\mu)$$

constant

- → REPLACE HIGHER BOUNCES WITH

$$\sum x^n = \left(\frac{1}{1-x}\right)$$

— REDUCE # FOURIER COEFFICIENTS NEEDED + # GAUSS POINTS (For angular integral) by TRUNCATING THE PHASE FUNCTION



- SCALE $\tilde{\omega}$ & τ TO TAKE ACCOUNT OF REMOVAL OF SOME OF SCATTERING X SECTION

$$\sigma_{scatt}^{T} = (1 - \Delta P)\, \sigma_{scatt}^{U}$$

$$\sigma_{abs}^{T} = \sigma_{abs}^{U}$$

$$\tilde{\omega}_0^{T} = \frac{(1-\Delta P)\, \tilde{\omega}_0^{U}}{(1 - \Delta P\, \tilde{\omega}_0^{U})}$$

$$\tau^{T} = (1 - \Delta P\, \tilde{\omega}_0^{U})\, \tau^{U}$$

where

$$\Delta P = \int (P^{U} - P^{T}) \frac{d\Omega}{4\pi}$$

- RENORMALIZE P^T

$$\longrightarrow \int P_R^{T} \frac{d\Omega}{4\pi} = 1$$

- USE P_R^T, $\tilde{\omega}_0^T$, τ^T IN 2X EQUATIONS $\longrightarrow$

$$I^{T}(\mu, \mu_0, \phi - \phi_0)$$

- CORRECT SINGLE SCATTERING (SS) PART OF I^T TO UNTRUNCATED SS:

for diffuse field

$$I^{C}(\mu, \mu_0, \phi - \phi_0) = I^{T}(\mu, \mu_0, \phi - \phi_0)$$

$$+ \; SS(\tau^{U}, P^{U}, \tilde{\omega}_0^{U}) - SS(\tau^{T}, P^{T}, \tilde{\omega}_0^{T})$$

Single scattering component for untruncated phase function

SS for truncated phase function

— RATIONALE

WHEN Θ SMALL, SS DOMINATES OVER MS (multiple scattering)

GAUSSIAN QUADRATURE

— **WHAT IS A QUADRATURE?**

replace integral by finite sum:

$$\int_a^b f(x)\,dx = \sum f(x_i)\, W_i$$

— **WHAT IS GAUSS' QUADRATURE?**

$$\int_{-1}^{1} f(\mu)\, d\mu = \sum_{j=1}^{m} f(\mu_j)\, a_j$$

where:

μ_j are the zeros of the Legendre polynomial of order m: $P_m(\mu)$

a_j is given by

$$a_j = \frac{1}{P_m'(\mu_j)} \int_{-1}^{1} \frac{P_m(\mu)\, d\mu}{(\mu - \mu_j)}$$

— **WHY IS THIS USEFUL?**

EXACTLY CORRECT FOR POLYNOMIALS IN μ OF DEGREE $\leq 2m-1$

(FOLLOWS FROM ORTHOGONALITY OF LEGENDRE POLYNOMIALS)

— **NOTES**

— For $\int_{-1}^{+1}$ SYMMETRY ABOUT ZERO

$\longrightarrow$ PAIRS OF μ : $\pm\mu_j$ $(j = 1 \to \frac{m}{2})$

WEIGHTS : a_j SAME FOR $\pm\mu_j$

— OTHER LIMITS OF INTEGRATION $\int_a^b$

For $\int_a^b \longrightarrow \left[\begin{array}{l} \tilde{a}_j = (b-a)\, a_j \\ \tilde{\mu}_j = \dfrac{(b+a)}{2} + \dfrac{(b-a)}{2}\,\mu_j \end{array}\right]$ For $\int_{-1}^{1}$

FEAUTRIER METHOD

WHAT IS IT?

AN ALTERNATIVE, ACCURATE NUMERICAL SOLUTION OF THE RT EQUATION

WHAT IS ITS BASIC

1. START WITH RT EQ. INVOLVING FIVE INDEPENDENT BASIC VARIABLES : $\tau, \mu, \mu_0, \phi, \phi_0$
2. $\int d\phi$ + FOURIER ANALYSIS $\rightarrow$ INDEPENDENT EQS INVOLVING τ, μ, μ_0
3. DISCRETIZE IN $\mu, \mu_0 \rightarrow$ 2nd ORDER DIFFERENTIAL EQUATION (MATRIX) IN τ
4. DISCRETIZE $\tau \rightarrow$ EQUATION IN BLOCK, TRIDIAGONAL FORM $\rightarrow$ INVERT TO OBTAIN SOLUTION

FUNDAMENTAL EQUATIONS

$$\mathbf{v}(\tau) = \mathbf{A}^{-1}(\tau)\left[\frac{d\mathbf{u}(\tau)}{d\tau} + \mathbf{v}_0(\tau)\right]; \tag{8.46}$$

substituting (8.46) into (8.39) yields

$$\frac{d}{d\tau}\left[\mathbf{A}^{-1}(\tau)\frac{d\mathbf{u}(\tau)}{d\tau}\right] - \mathbf{B}(\tau)\mathbf{u}(\tau) = -\frac{d}{d\tau}[\mathbf{A}^{-1}(\tau)\mathbf{v}_0(\tau)] - \mathbf{u}_0(\tau). \tag{8.47}$$

Since $\mathbf{A}(\tau)$ and $\mathbf{B}(\tau)$ are positive definite matrices, (8.47) can be interpreted as the multicomponent generalization of the steady-state diffusion equation. ~~For isotropic scattering in an atmosphere illuminated~~

where

$$\mathbf{u}(\tau) = \tfrac{1}{2}[\mathbf{I}^+(\tau) + \mathbf{I}^-(\tau)], \tag{8.40}$$

$$\mathbf{v}(\tau) = \tfrac{1}{2}[\mathbf{I}^+(\tau) - \mathbf{I}^-(\tau)], \tag{8.41}$$

$$\mathbf{u}_0(\tau) = \tfrac{1}{2}[\boldsymbol{\sigma}^+(\tau) - \boldsymbol{\sigma}^-(\tau)], \tag{8.42}$$

$$\mathbf{v}_0(\tau) = \tfrac{1}{2}[\boldsymbol{\sigma}^+(\tau) + \boldsymbol{\sigma}^-(\tau)], \tag{8.43}$$

SOURCE FUNCTIONS

and the elements of the $n \times n$ matrices are [see (8.6), (8.19) and (8.20) for definitions]

$$A_{ij}(\tau) = M_{ij}^+(\tau) - M_{ij}^-(\tau)$$
$$= \frac{1}{\xi_i}\{\delta_{ij} - \tfrac{1}{2}\gamma a_j[\rho(\tau;\xi_i,\xi_j) - \rho(\tau;\xi_i,-\xi_j)]\}, \tag{8.44}$$

$$B_{ij}(\tau) = M_{ij}^+(\tau) + M_{ij}^-(\tau)$$
$$= \frac{1}{\xi_i}\{\delta_{ij} - \tfrac{1}{2}\gamma a_j[\rho(\tau;\xi_i,\xi_j) + \rho(\tau;\xi_i,-\xi_j)]\}. \tag{8.45}$$

$\xi_i \Longleftrightarrow \mu_i$

$a_j \Longleftrightarrow$ GAUSS WEIGHT

$p \Longleftrightarrow$ PHASE FUNCTION

$\gamma \Longleftrightarrow \frac{1}{2}(1 + \delta_{0m})$ order of fourier coefficient

$\int p \frac{d\Omega}{4\pi} = \tilde{\omega}_0$

most direct path wasn't used because used to be too slow

NOTES

- HAS PROVED TO BE PRACTICAL ONLY RECENTLY WITH FAST MATRIX INVERSIONS ON COMPUTERS
- FASTER THAN 2X/+ (now)
- WE HAVE A WELL DOCUMENTED CODE FROM WARREN WISCOMBE (SEE BRIAN TOON)

APPLICATION OF DOUBLING/ADDING METHOD:
DERIVATION OF PROPERTIES OF MARTIAN DUST

DATA

IMAGES OF MARTIAN SKY OBTAINED BY VIKING LANDER CAMERAS

PROPERTIES TO BE RETRIEVED

(width parameter of size distrib)

- SIZE DISTRIBUTION (~~REA~~ ACTUALLY r_{eff} + v_{eff})
  X section weighted size; width
- SINGLE SCATTERING ALBEDO
- NON SPHERICAL PARAMETERS

APPROACH

— SHAPE OF $I/F(\Theta)$ AT SMALL Θ (10–30°) $\longrightarrow$ DIFFRACTION PEAK $\longrightarrow$ r_{eff} + v_{eff}

— ABSOLUTE $I/F(\Theta)$ AT $\Theta \sim 30-50°$ $\longrightarrow \tilde{\omega}_0(\lambda)$ (know τ from direct sun obs)

— SHAPE OF $I/F(\Theta)$ AT INTERMEDIATE + LARGE Θ $\longrightarrow$ NONSPHERICAL PARAMETERS

PRACTICAL TIPS

— USE PHASE FUNCTION TRUNCATION
$\longrightarrow$ MUCH FEWER FOURIER COEFF, GAUSS PTS

— TEST # FOURIER COEFF, GAUSS PTS.

TABLE 2. Number of Local Dust Storms Observed From Viking Orbiter 2

L_s	Period, VO 2 Revolutions	Description	Number of Local Dust Storms	Number/100 Revolutions
177–206	131–180	pre-dust storm 1	5	10
206–244	180–240	rapid decay phase, storm 1	2	3.3
244–274	240–285	slow decay phase, storm 1	4	8.9
274–327	285–374	rapid decay phase, storm 2	1	1.1

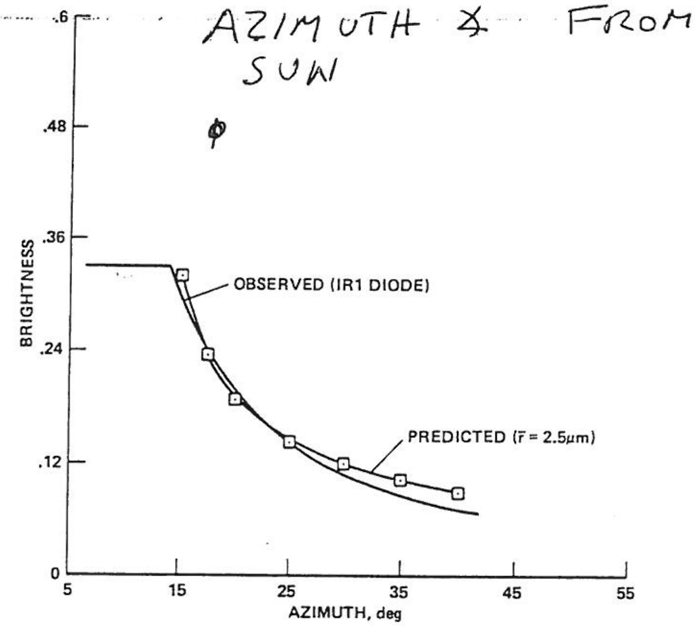


Fig. 5. Same type of comparison as in Figure 4, but here the observed data were taken with the IR1 diode having an effective wavelength of 0.87 μm (camera event 21 B 221).

and backward hemispheres (it equals 0 for isotropic scattering and 1 for totally forward scattering), and Q_e^s is the ratio of the interaction cross section to the geometric cross section.

All the above dust particle properties can be derived from analyses of the Viking lander imaging observations of the Martian sky and the sun. According to paper 1, the dust particles were uniformly mixed within the bottom few scale heights of the atmosphere during the primary portion of the mission. It seems likely that they were also uniformly mixed at other times, although further analysis is required to confirm this hypothesis.

The optical depths, which were discussed in the preceding section, are characterized by an effective wavelength of 0.67 μm. At other wavelengths, the optical depth scales linearly with Q_e. For the particle sizes of interest, Q_e at 0.67 μm lies within a few percent of the value of Q_e averaged over the solar spectrum. Thus, we can equate the measured value of τ with its value averaged over the solar spectrum. As was mentioned above, it is best to use the P.M. values of optical depth to characterize the dust component since, at least during the primary mission, the A.M. values have a nonnegligible contribution from a ground fog.

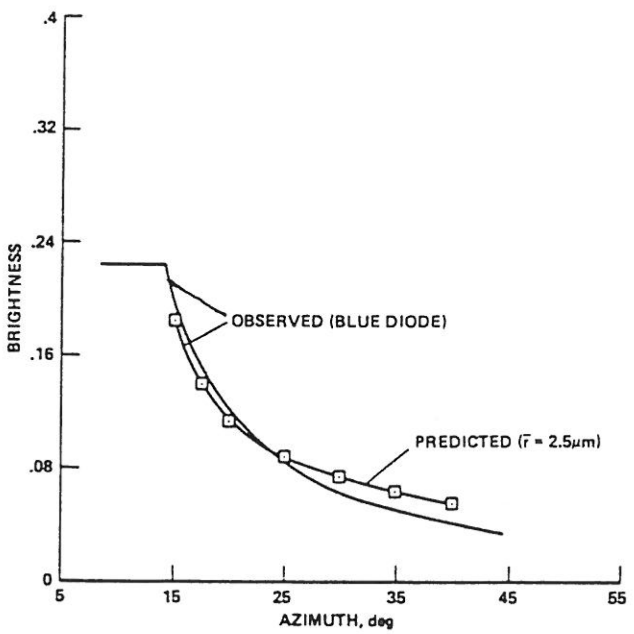


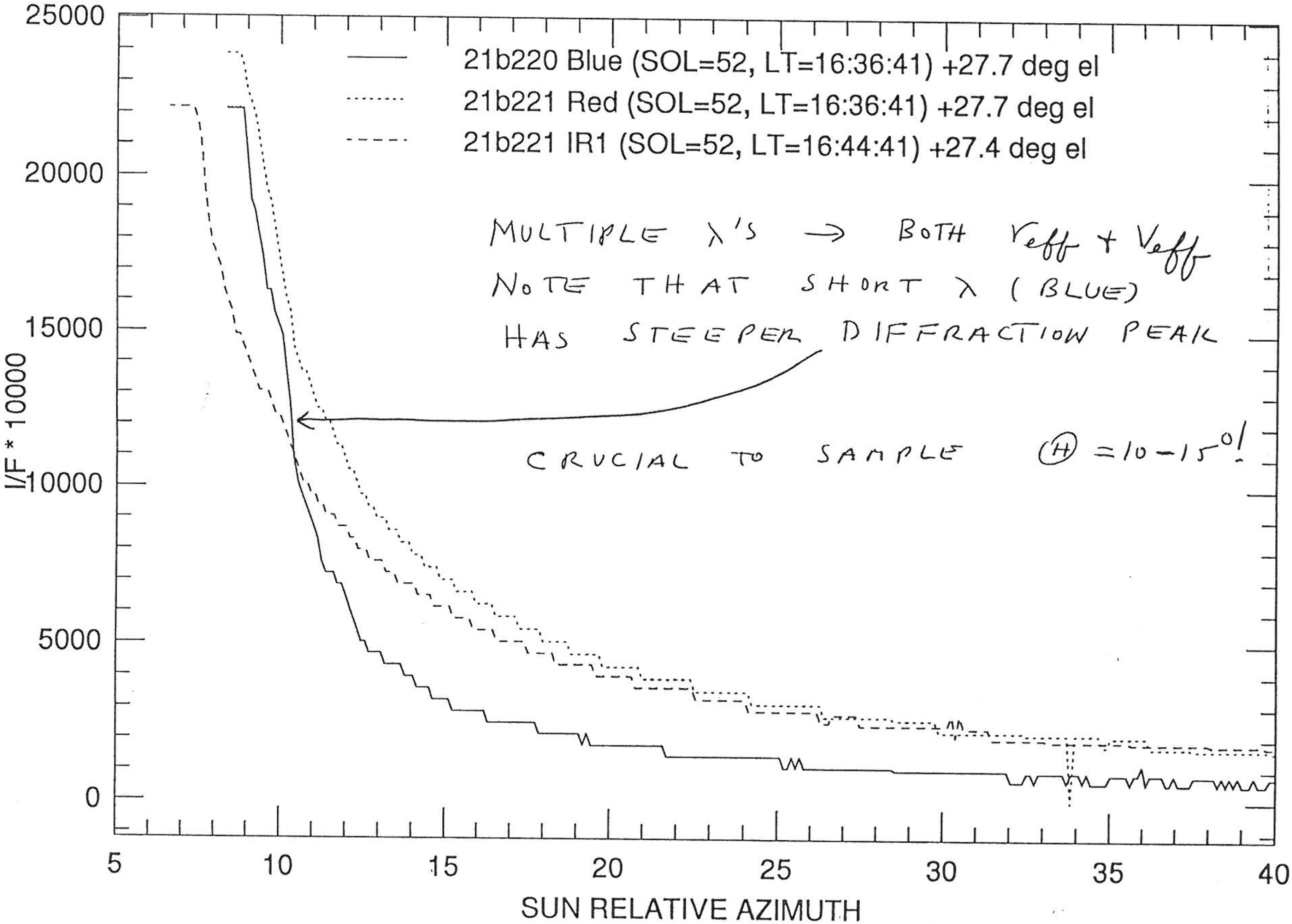
Fig. 4. Comparisons between the observed angular variation of sky brightness and that of a theoretical model characterized by a modified gamma function with $\alpha = 2$, $\gamma = \frac{1}{2}$, and $r_m = 0.4$ μm. The angular coordinate is the azimuthal distance from the sun in degrees. The observed curve was derived from an image taken at VL 2 on sol 52 with a blue diode having an effective wavelength of 0.49 μm (camera event 21 B 220). The theoretical curve was normalized to agree with the observed value at an azimuth of 25°.

The remaining properties—$\bar{\omega}_0$, $\langle\cos\theta\rangle$, and Q_e^s—can be found once the particles' size distribution, shape, and optical constants are known. As was discussed in paper 1, the latter set of parameters can be derived from the photometric characteristics of the Martian sky, as observed with the lander cameras. For this purpose, a numerically accurate computer program is used that incorporates a newly developed method for treating scattering by nonspherical particles and that allows for multiple scattering within the aerosol layer and the radiative interaction of this layer with the surface.

Preliminary estimates of the cross section average mean particle radius, $\bar{r}$, several shape factors ALFO and FTB, and the imaginary index of refraction n_i at wavelengths ranging from 0.5 to 1.0 μm were obtained in paper 1 from an analysis of primary mission pictures. In accord with the properties of commonly abundant silicates, the real index of refraction, n_r, was assumed to equal 1.5 and be independent of wavelength in the visible and near infrared. Unfortunately, analysis of additional pictures from this primary mission have shown that the estimated value for $\bar{r}$ of 0.4 μm leads to certain inconsistencies. While such a value for $\bar{r}$ gives a good match to the angular variation of the sky brightness close to the sun for pictures taken with a blue filter, it gives an unsatisfactory fit to analogous data obtained with red and near-infrared filters. Larger values of $\bar{r}$ are required to fit these longer wavelength results.

The above problem can be attributed to their existing two possible solutions for the value of $\bar{r}$ for the blue picture analyzed in paper 1. One solution is $\bar{r} = 0.4$ μm, while the second solution is an $\bar{r}$ of several microns. By analyzing data at several wavelenths and/or over a larger range of angles close to the sun, we can resolve this ambiguity. As was noted above, the smaller value of $\bar{r}$ provides an unsatisfactory fit to data at other wavelengths. For the purpose of examining further the larger value of $\bar{r}$, we have chosen to represent the particle size distribution function, $n(r)$, with a modified gamma function:

$$n(r) = cr^{\alpha} \exp\left[-(\alpha/\gamma)(r/r_m)^{\gamma}\right] \quad (1)$$


21b220 Blue (SOL=52, LT=16:36:41) +27.7 deg el
21b221 Red (SOL=52, LT=16:36:41) +27.7 deg el
21b221 IR1 (SOL=52, LT=16:44:41) +27.4 deg el
MULTIPLE λ'S → BOTH r_{eff} + v_{eff}
NOTE THAT SHORT λ (BLUE)
HAS STEEPER DIFFRACTION PEAK
CRUCIAL TO SAMPLE Θ = 10 - 15°!
I/F * 10000
25000
20000
15000
10000
5000
0
5
10
15
20
25
30
35
40
SUN RELATIVE AZIMUTH

NONSPHERICAL PARAMETERS

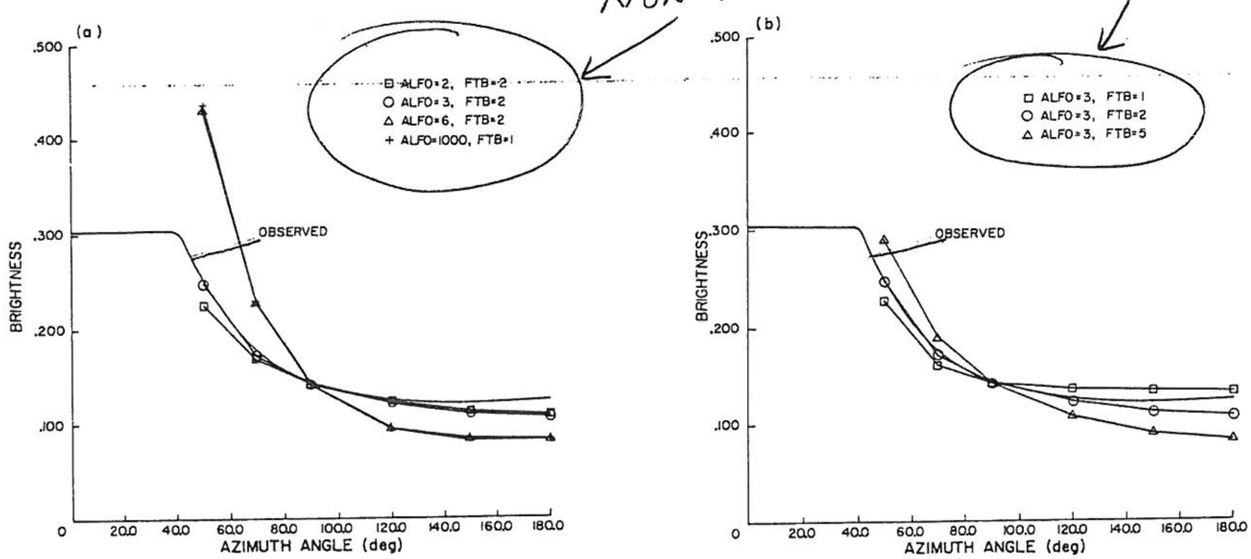


Fig. 9. (*a*) Comparison between the observed angular variation of sky brightness and that predicted by models having varying values of the parameter *ALFO*. The observed curve was obtained from the panorama picture taken at VL-1 on sol 0 at 1619 hours with the survey diode (camera event 12A002). At that time the sun's elevation angle was 36.8° in local horizon coordinates. The line scan shown in this figure refers to an elevation angle of view equal to 15°. Azimuth angles are measured from the azimuthal position of the sun at the time of the picture. For the exact location of the picture, refer to *Mutch* [1977, Figure 1]. All theoretical curves were computed with $RM = 0.4\ \mu m$ and $FTB = 2$. They have been normalized to agree with the observed value at an azimuth of 95°. (*b*) Comparison between the same observed curve shown in Figure 9*a* and the sky brightness values computed for aerosol models having varying values of *FTB*. All models have $RM = 0.4\ \mu m$ and $ALFO = 2$. The theoretical curves have been normalized to agree with the observed value at an azimuth of 95°.

sition from an analysis of the wavelength dependence of the sky brightness. For this purpose we used pictures taken close together in time with some or all of the six narrow band diodes. These six diodes are the blue, green, red, and three near-infrared channels (IR1, IR2, and IR3), which span the spectral region from about 0.4–1.1 μm [*Huck et al.*, 1977]. In performing this analysis we used values derived in the last section for most of the aerosol and ground properties. The only free parameters now are the values of the imaginary index of refraction of the aerosols n_i and the geometric albedo of the surface B_0. These are the two parameters that should vary the most with wavelength. They are determined in an iterative manner along the lines of the procedure used in the previous section. Note that the analysis carried out in the last section yielded n_i and B_0 only for the blue and survey diodes.

The analysis discussed above leads to estimates of n_i for each of the six narrow band channels. These values are inserted into a computer program analogous to the one described by *Park and Huck* [1976] to obtain a coarse resolution plot of n_i as a function of wavelength. In effect, this program removes the overlap in the wavelength coverage of the different channels. To obtain the desired information about the aerosols' composition, we compare the absolute value of n_i and its wavelength dependence with values that characterize plausible candidate materials.

In addition to the above analyses we also used color pictures to estimate the ratio of the red to blue brightness of the sky. Such data are useful in a relative sense. Diurnal and seasonal changes in sky color provide clues concerning the compositional changes that accompany optical depth variations.

Results

We obtained estimates of n_i and B_0 from an analysis of the absolute value of the sky and ground brightnesses found from images obtained on sol 39 at VL-1. Commencing at approximately 1311 hours, pictures were taken in quick succession with each of the narrow band diodes. At this time the sun's elevation angle was approximately 77.7°. The absolute brightness values of the sky were found by averaging a 20 × 20 array of samples centered at an elevation angle of 12° and an azimuth of 26° from the sun's azimuth, while a similar averaging was used at elevation and azimuthal angles of −7° and 54°, respectively, to find absolute brightness values for the surface.

Figure 10 illustrates the manner in which n_i was determined for the blue diode. The straight horizontal line shows the observed value of the sky brightness in the usual reflectance units. The curved line shows the theoretical dependence of the

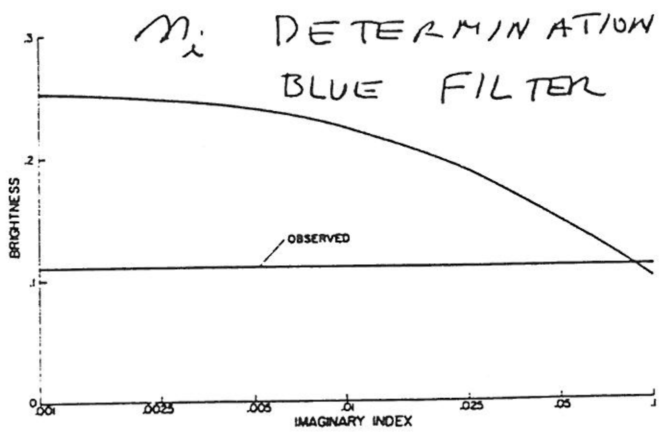


Fig. 10. Illustration of the procedure used to find the imaginary index of refraction n_i. The horizontal line shows the observed value of sky brightness, in reflectance units, found from a picture taken at VL-1 with the blue diode on sol 39 (camera event 12B069). The observed value refers to a position located at an elevation angle of 20° and an azimuthal distance of 26° from the sun. At the time of the picture, 1311 hours, the sun was at an elevation angle of 77.7°. For the exact location of the picture, refer to *Mutch* [1977, Figure 1]. The other curve of this figure shows the theoretical dependence of sky brightness at this position as a function of n_i. The inferred value of n_i is found from the intersection of the theoretical curve with the horizontal line.

READING LIST

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